

Phase properties of the superposition of squeezed and displaced number states

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The phase properties of superposition of squeezed and displaced number states are examined in the framework of Pegg-Barnett formalism. Moreover, amplitude squeezing and phase squeezing are discussed.

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I. INTRODUCTION

In classical optics, the concepts of the intensity and phase of optical fields have a well-defined meaning. That is the electromagnetic field (E) associated with one mode, $E = A \exp(i\theta)$, has a well defined amplitude (A) and phase (θ). This is not so simple in quantum optics where the mean photon number and the phase are represented by non-commuting operators and consequently they cannot be defined well simultaneously. In fact, the concept of phase is a controversial problem from the earlier days of quantum optics [1, 2]. In general there are three methods of treating this issue [3]. The first one considers the phase as a physical quantity in analogy to position or momentum by representing it with a linear Hermitian phase operator. The second one involves c-number variables (real or complex) in phase spaces or their associated distribution functions, or ensembles of trajectories. The third one is the operational phase approach in which the phase information is inferred from the experimental data by analogy with the classical analysis of the experiment. Each approach has advantage and disadvantage points. Indeed, the interest in the phase properties

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has been motivated by experimental realization of optical homodyne tomography [4] allowing quantum phase mean values to be calculated from the measured field density matrix.

As is well known squeezed states of light (i.e. the states of light with reduced fluctuations in one quadrature below the level associated with the vacuum state [5]) have become a subject of intensive studies owing to their interesting applications in various devices, e.g. in optical communication systems [6], interferometric techniques [7], and in an optical waveguide tap [8]. Moreover, there are a number of successful experiments producing such light [9]. These states have phase sensitive noise properties and therefore several works have been devoted to follow such properties, e.g. for single-mode squeezed states [10, 11]. On the other hand, the superposition principle is at the heart of quantum mechanics. It implies that probability densities of observable quantities usually exhibit interference effects instead of simply being added. The most significant example reflecting the power of such principle are the Schrödinger cat states [12], which exhibit various nonclassical properties, even if the original states are close to the classical states [13]. Based on this principle, a general class of quantum states has been introduced as a superposition of displaced and squeezed number states [14, 15]. It is important to point out the basis of this class are the squeezed displaced number states [16] which are purely nonclassical states. This class is represented as a single mode vibration of the electromagnetic field suddenly squeezed-plus-displaced by a collection of two displacements 180° out of phase with respect to each other [14, 15], i.e.

$$|r, \alpha, n\rangle_\epsilon = \lambda_\epsilon [\hat{D}(\alpha) + \epsilon \hat{D}(-\alpha)] \hat{S}(r) |n\rangle, \quad (1)$$

where λ_ϵ is the normalization constant, $\hat{D}(\alpha)$ and $\hat{S}(r)$ are displacement and squeeze operator respectively, while α and r are displacement and squeeze parameters; $\epsilon = |\epsilon|e^{i\phi}$ is a complex parameter, and $|n\rangle$ denotes a Fock state. Squeeze and displacement operators are given, respectively, by

$$\hat{S}(r) = \exp\left[\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})\right], \quad (2)$$

$$\hat{D}(\alpha) = \exp(\hat{a}^\dagger \alpha - \hat{a} \alpha^*), \quad (3)$$

where \hat{a} and \hat{a}^\dagger are annihilation and creation operators. The normalization constant λ_ϵ is given by

$$|\lambda_\epsilon^2|^{-1} = 1 + |\epsilon|^2 + 2|\epsilon| \exp(-2|t|^2) L_n(4|t|^2) \cos \phi, \quad (4)$$

with

$$t = \alpha \cosh r + \alpha^* \sinh r, \quad (5)$$

and $L_n(\cdot)$ is the Laguerre polynomial. The density matrix $\hat{\rho}$ of this state can be written as

$$\hat{\rho} = |\lambda_\epsilon|^2 (\hat{\rho}_M + \hat{\rho}_I). \quad (6)$$

The part of the density matrix corresponding to the statistical mixture of two squeezed displaced number states is:

$$\hat{\rho}_M = \hat{D}(\alpha) \hat{S}(r) |n\rangle \langle n| \hat{S}^\dagger(r) \hat{D}^\dagger(\alpha) + |\epsilon|^2 \hat{D}(-\alpha) \hat{S}(r) |n\rangle \langle n| \hat{S}^\dagger(r) \hat{D}^\dagger(-\alpha), \quad (7)$$

while the quantum interference part has the form

$$\hat{\rho}_I = \epsilon^* \hat{D}(\alpha) \hat{S}(r) |n\rangle \langle n| \hat{S}^\dagger(r) \hat{D}^\dagger(-\alpha) + \epsilon \hat{D}(-\alpha) \hat{S}(r) |n\rangle \langle n| \hat{S}^\dagger(r) \hat{D}^\dagger(\alpha). \quad (8)$$

This quantum interference part of the density matrix contains information about the quantum interference between component states $\hat{D}(\pm\alpha) \hat{S}(r) |n\rangle$ and this will be responsible for some interesting behaviour of the phase distribution, as we will see.

For completeness, the physical interpretation of such states can be related to a superposition of coherent states formed due to two excitations on particularly excited harmonic oscillators [16]. It is clear that these states enable us to obtain generalizations of some results given in the literature. It has been shown that these states can be generated, by means of the so-called quantum state engineering, and also by means of trapping ions (for more details, see refs.[14, 15]). The quantum properties of these states reveal that they can exhibit sub-Poissonian statistics, quadrature squeezing and oscillations in photon-number distribution. Moreover, the influence of thermal noise on the behaviour of such superposition has been considered [17] showing that the correlation between different oscillators is essentially responsible for nonclassical effects similar to Schrödinger cat states. This fact has been demonstrated from the behaviour of Wigner function and photon-number distribution.

In this article we study the phase properties of the superposition (1) using Pegg-Barnett technique [18] which is most convenient for the current problem. This investigation is organized as follows: In section 2 we give the description of the Pegg-Barnett technique and the

basic relations related to the state (1), followed by section 3 where the results are discussed. The conclusions are summed in section 4.

II. BASIC RELATIONS

Here we give essential background for Pegg-Barnett [18] phase formalism and the basic relations for the state under discussion. Their formalism is based on introducing a finite $(s + 1)$ -dimensional space Ψ spanned by the number states $|0\rangle, |1\rangle, \dots, |s\rangle$. The physical variables (expectation values of Hermitian operators) are evaluated in the finite dimensional space Ψ and at the final stage the limit $s \rightarrow \infty$ is taken. A complete orthonormal basis of $s + 1$ states is defined on Ψ as

$$|\Theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{k=0}^s \exp(ik\Theta_m) |k\rangle, \quad (9)$$

where

$$\Theta_m = \Theta_0 + \frac{2\pi m}{s+1}, \quad m = 0, 1, \dots, s. \quad (10)$$

The value of Θ_0 is arbitrary and defines a particular basis of $s + 1$ mutually orthogonal states. The Hermitian phase operator is defined as

$$\hat{\Phi}_\theta = \sum_{m=0}^s \Theta_m |\Theta_m\rangle \langle \Theta_m|, \quad (11)$$

where the subscript shows the dependence on the choice of Θ_0 . The phase states (9) are eigenstates of the phase operator (11) with the eigenvalues Θ_m restricted to lie within a phase window between Θ_0 and $2\pi + \Theta_0$. The expectation value of the phase operator (11) in a pure state $|\psi\rangle = \sum_{m=0}^\infty C_m |m\rangle$, where C_m is the weighting coefficient including the normalization constant, is given by

$$\langle \psi | \hat{\Phi}_\theta | \psi \rangle = \sum_{m=0}^s \Theta_m |\langle \psi | \Theta_m \rangle|^2. \quad (12)$$

The density of phase states is $(s+1)/(2\pi)$, so the continuum phase distribution as s tends to infinity is

$$P(\Theta) = \lim_{s \rightarrow \infty} \frac{s+1}{2\pi} |\langle \Theta_m | \psi \rangle|^2 \quad (13)$$

$$= \frac{1}{2\pi} \sum_{m,m'=0}^{\infty} C_m C_{m'}^* \exp[i(m - m')\Theta],$$

where Θ_m has been replaced by the continuous phase variable Θ . As soon as the phase distribution $P(\Theta)$ is known, all the quantum-mechanical phase moments can be obtained as a classical integral over Θ . The phase distribution is normalized such as

$$\int_{-\pi}^{\pi} P(\Theta) d\Theta = 1. \quad (14)$$

One of the particular interesting quantities in the description of the phase is the phase variance determined by

$$\langle (\Delta \hat{\Phi})^2 \rangle = \int \Theta^2 P(\Theta) d\Theta - \left(\int \Theta P(\Theta) d\Theta \right)^2. \quad (15)$$

As we mentioned earlier the mean photon number and the phase are conjugate quantities in this approach and consequently they obey the following uncertainty relation

$$\langle (\Delta \hat{\Phi})^2 \rangle \langle (\Delta \hat{n})^2 \rangle \geq \frac{1}{4} |\langle [\hat{n}, \hat{\Phi}] \rangle|^2. \quad (16)$$

The number–phase commutator appearing on the right hand side of (16) can be calculated for any physical state [18] as

$$\langle [\hat{n}, \hat{\Phi}] \rangle = i[1 - 2\pi P(\Theta_0)]. \quad (17)$$

In relation to (16), we can give the notion of the number and phase squeezing [19, 20] through the relation

$$S_n = \frac{\langle (\Delta \hat{n})^2 \rangle}{\frac{1}{2} |\langle [\hat{n}, \hat{\Phi}] \rangle|} - 1, \quad (18)$$

$$S_\theta = \frac{\langle (\Delta \hat{\Phi})^2 \rangle}{\frac{1}{2} |\langle [\hat{n}, \hat{\Phi}] \rangle|} - 1. \quad (19)$$

The values of -1 of these equations means maximum squeezing of the photon number or the phase.

We shall use above relations to study the phase distribution for the superposition of displaced and squeezed number states (1). In this case $C_m = C_m(r, \alpha, n, \epsilon) = \langle m|r, \alpha, n \rangle_\epsilon$ and this quantity can be calculated through the identity

$$\langle m|n, \alpha, r \rangle_\epsilon = \int_{-\infty}^{+\infty} dx \Upsilon_m^*(x) \Upsilon_n^{(\epsilon)}(x, r, \alpha), \quad (20)$$

where $\Upsilon_n^{(\epsilon)}(x, r, \alpha) = \langle x|n, r, \alpha \rangle_\epsilon$ is the wavefunction of (1) having the form [15]

$$\begin{aligned} \Upsilon_n^{(\epsilon)}(x, r, \alpha) &= \frac{\lambda_\epsilon e^{\frac{\tau}{2}}}{\sqrt{2^n n!}} \left(\frac{\omega}{\pi \hbar}\right)^{\frac{1}{4}} \left\{ \exp \left[-\frac{e^{2r}}{2} \left(x \sqrt{\frac{\omega}{\hbar}} - \sqrt{2\alpha} \right)^2 \right] \right. \\ &\quad \times H_n \left[e^r \left(x \sqrt{\frac{\omega}{\hbar}} - \sqrt{2\alpha} \right) \right] + \epsilon \exp \left[-\frac{e^{2r}}{2} \left(x \sqrt{\frac{\omega}{\hbar}} + \sqrt{2\alpha} \right)^2 \right] \\ &\quad \left. \times H_n \left[e^r \left(x \sqrt{\frac{\omega}{\hbar}} + \sqrt{2\alpha} \right) \right] \right\}, \end{aligned} \quad (21)$$

where α is real, $H_n(\cdot)$ is the Hermite polynomial of order n ; ω and \hbar are frequency of the harmonic oscillator and Planck's constant divided by 2π . Substituting from (21) together with the wavefunction $\Upsilon_m(x)$ of the Fock state (which can be deduced from (21) by simply setting $r = \alpha = \epsilon = 0$) into (20) and carrying out the integration, using the identity [21]

$$\begin{aligned} &\sqrt{\frac{M}{\pi}} \int_{-\infty}^{+\infty} dx H_m(x) H_n(\Lambda x + d) \exp(-Mx^2 + cx) = \exp\left(\frac{c^2}{4M}\right) \\ &\quad \times \left(\sqrt{\frac{M-1}{M}}\right)^m \left(\sqrt{\frac{M-\Lambda^2}{M}}\right)^n \sum_{j=0}^{\min(m,n)} \frac{n!m!}{j!(n-j)!(m-j)!} \\ &\quad \times \left(\frac{2\Lambda}{\sqrt{(M-1)(M-\Lambda^2)}}\right)^j H_{m-j}\left(\frac{c}{2\sqrt{(M-1)M}}\right) H_{n-j}\left(\frac{c\Lambda+2dM}{2\sqrt{M(M-\Lambda)}}\right), \end{aligned} \quad (22)$$

we arrive at [15]

$$\begin{aligned} \langle m|n, \alpha, r \rangle_\epsilon &= \frac{\lambda_\epsilon \left(\frac{\tanh r}{2}\right)^{\frac{m}{2}}}{\sqrt{n!m!} \cosh r} \exp\left[\frac{\tau^2}{2}(\tanh r - 1)\right] \sum_{j=0}^{\min(m,n)} \frac{n!m!}{j!(n-j)!(m-j)!} \left[\frac{2}{\sqrt{\sinh 2r}}\right]^j \\ &\quad \times \left[\frac{-\tanh r}{2}\right]^{\frac{(n-j)}{2}} H_{n-j}\left(\frac{i\alpha}{\sqrt{\sinh 2r}}\right) H_{m-j}\left(\frac{\tau}{\sqrt{\sinh 2r}}\right) [1 + (-1)^{(n+m)}\epsilon], \end{aligned} \quad (23)$$

where $\tau = \alpha \exp(r)$. Another way for the derivation of (23) can be found in the appendix of [22]. When $r \rightarrow 0$, (23) reduces to the distribution coefficient for the superposition of displaced Fock states as

$$\begin{aligned}
C_m(r=0, \alpha, n, \epsilon) &= \lambda_\epsilon \sqrt{n!m!} \exp(-\frac{\alpha^2}{2}) \sum_{j=0}^{\min(m,n)} \frac{(-1)^{n-j} \alpha^{n+m-2j}}{j!(n-j)!(m-j)!} [1 + (-1)^{n+m} \epsilon] \\
&= \lambda_\epsilon \left[\frac{p!}{q!} \right]^{\frac{1}{2}} (-1)^{n-p} \alpha^{n+m-2p} \exp(-\frac{\alpha^2}{2}) [1 + (-1)^{n+m} \epsilon] L_p^{q-p}(\alpha^2),
\end{aligned} \tag{24}$$

where $L_n^\gamma(\cdot)$ is the associated Laguerre polynomial, and $p = \min(n, m)$ and $q = \max(n, m)$. The photon number variance is defined as

$$\begin{aligned}
\langle (\Delta \hat{n})^2 \rangle &= \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 \\
&= \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle + \langle \hat{n} \rangle - \langle \hat{n} \rangle^2,
\end{aligned} \tag{25}$$

where the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. In terms of (1) the quantities $\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle$ and $\langle \hat{n} \rangle$ can be straightforwardly calculated, however, the explicit forms can be found in [15]. For completeness, the phase variance of (1) reads

$$\begin{aligned}
\langle (\Delta \hat{\Phi})^2 \rangle &= \frac{\pi^2}{3} + 4 \text{Re} \sum_{m > m'} C_m(r, \alpha, n, \epsilon) C_{m'}^*(r, \alpha, n, \epsilon) \frac{(-1)^{m-m'}}{(m-m')^2} \\
&\quad - 4 \left[\text{Rei} \sum_{m > m'} C_m(r, \alpha, n, \epsilon) C_{m'}^*(r, \alpha, n, \epsilon) \frac{(-1)^{m-m'}}{(m-m')^2} \right]^2.
\end{aligned} \tag{26}$$

The value $\pi^2/3$ is the phase variance for a state with uniformly distributed phase, e.g. the vacuum state. Finally, for the future purpose, we give the form of Wigner function of (1) [14, 15]

$$\begin{aligned}
W^{(\epsilon)}(x, y) &= \frac{2(-1)^n |\lambda_\epsilon|^2}{\pi} \{ |\epsilon|^2 \exp[-2(y^2 e^{-2r} + e^{2r}(x + \alpha)^2)] \\
&\quad \times L_n[4(y^2 e^{-2r} + e^{2r}(x + \alpha)^2)] + \exp[-2(y^2 e^{-2r} + e^{2r}(x - \alpha)^2)] \\
&\quad \times L_n[4(y^2 e^{-2r} + e^{2r}(x - \alpha)^2)] + 2 \exp[-2(y^2 e^{-2r} + e^{2r}x^2)] \\
&\quad \times |\epsilon| L_n[4(y^2 e^{-2r} + e^{2r}x^2)] \cos(4y\alpha - \phi) \}.
\end{aligned} \tag{27}$$

Based on the results of the present section we discuss the phase properties of the state (1) in the following section.

III. DISCUSSION OF THE RESULTS

For understanding the behaviour of the phase distribution of the states (1) it is more convenient firstly to study such properties for two subsidiary states which are the superposition of displaced number states [23], and squeezed and displaced number states [16]. These two states are of interest because it has been shown for the former states that they can exhibit strong sub-Poissonian character as well as quadrature squeezing. The bunching and antibunching properties for the latter states have been discussed in [24]. Investigation of such properties for these two kinds of states most clearly illustrates the role of the different parameters in the state (1) in the behaviour of the distribution.

In the following two subsections we discuss the phase distribution determined by (13), and the phase variance and phase squeezing, respectively. For simplicity we restrict our investigation to real values of α and ϵ .

A. Phase probability distribution

In general we find that there are two regimes controlling the behaviour of the phase distribution for the states under discussion which are $\alpha \gg 1$ and $\alpha \leq 1$ provided that n is finite. For the first case, the even- and odd-cases (i.e. $\epsilon = 1$ and -1) provide similar behaviours and there is one-to-one correspondence between the components of density matrix (6) and that corresponding to the curves. In other words, $\hat{\rho}_M$ is responsible for the central behaviour (i.e. around $\Theta = 0$) of the phase distribution and $\hat{\rho}_I$ is responsible for the lateral behaviour, i.e. when $\Theta \rightarrow \pm\pi$ (see Fig. 2a). Nevertheless, all these features are washed out in the second regime and the behaviour dramatically change. Such behaviours can be understood well if we turn our attention to the behaviour of the W -function where the interference in phase space may be seen clearly. More illustratively, W -function (27) includes three terms, the first two terms representing the statistical mixture of squeezed displaced number states and the third one is the interference part. When $\alpha \leq 1$ (as shown in Fig. 1a and b for superposition of displaced number states) the contributions of these components are comparable so that even- and odd-cases distributions have different shapes. On the other hand, when $\alpha \gg 1$ the structures of the W -function for even- and odd-cases are almost similar, i.e. they include two symmetrical peaks originated at $\pm\alpha$ and interference

fringes are in between. Of course, such situation is still valid if squeezing in the displaced superimposed number state in optical cavity is considered, however, the peaks will be then stretched. This behaviour is reflected in the behaviour of the phase distribution, as we will see. It should be stressed that the greatest degree of the nonclassical behaviour for the superposition (1) actually occurs for rather low values of α [14, 15].

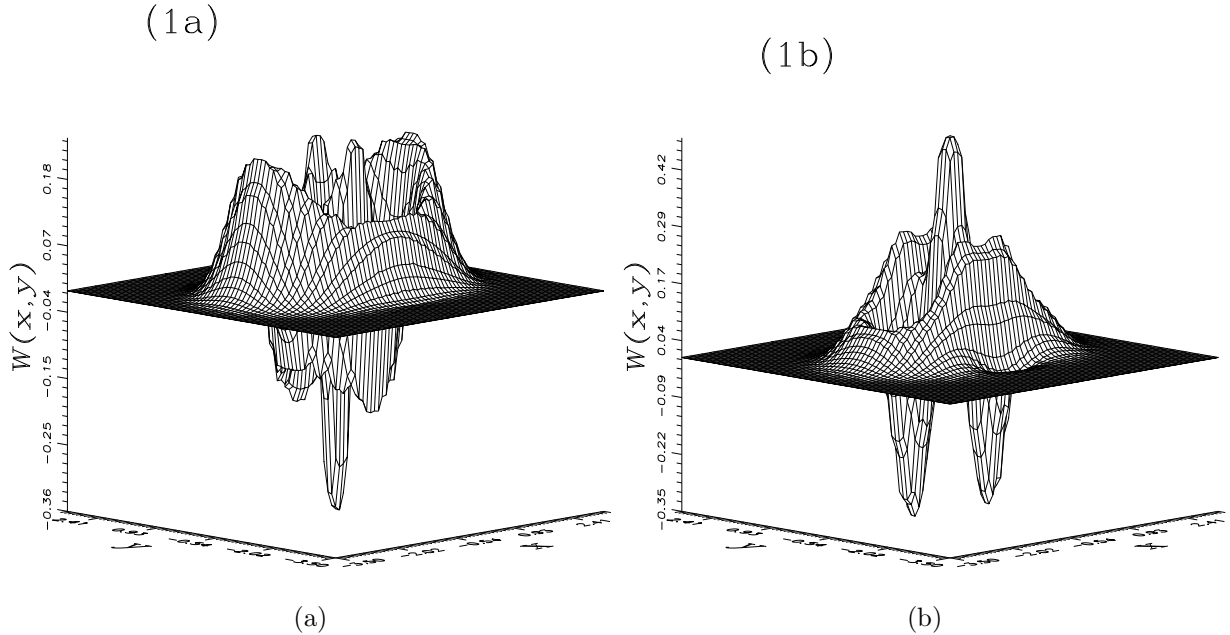


FIG. 1: W -function against x and y for $n = 1, \alpha = 1$ and for a) $\epsilon = 1$; b) $\epsilon = -1$.

We start our discussion by focusing the attention on the behaviour of the superposition of displaced number state. It is important to mention that the properties of displaced number states $\hat{D}(\alpha)|n\rangle$ have been given in [25] and their phase properties in [26]. In fact the phase properties of such states are interesting since they connect the number state, which has no phase information, and coherent states, which play the boundary role between the classical and nonclassical states and always exhibit single peak structure of phase distribution. The phase distribution of displaced number states exhibits nonclassical oscillations with the number of peaks which are equal to $n + 1$. This result is interpreted in terms of the area of overlap in phase space [26]. Unfortunately, the situation is completely different for the superposition of displaced number state (see Fig. 2a and b for shown values of the parameters). From Fig. 2a it is clear that the quantum interference between component states $\hat{D}(\alpha)|n\rangle$ and $\hat{D}(-\alpha)|n\rangle$ leads to the lateral nonclassical oscillations which become more pronounced and narrower as n increases. However, the statistical mixture of displaced number

states shows $n + 1$ peaks around $\Theta = 0$. We have not presented the phase distribution for the odd-case since it is similar to that of the even-case. Comparing this behaviour with that of even (or odd) coherent states, we conclude that the number states in the superposition make the phase information more significant [1]. Turning our attention to the Fig. 2b where $\alpha = 1$, we can see that the distribution of displaced number states exhibits two-peak structure as expected for $n = 1$ [26], however, the distribution of even-case displays one central peak at $\Theta = 0$ and two wings as $\Theta \rightarrow \pm\pi$, and finally the distribution of odd-case provides four peaks. That is the distribution here is irregular, however, more smoothed than before and the structure of the statistical mixture (central part) is modified by the action of the interference term arising from the superposition of the states; this was the case for W -function.

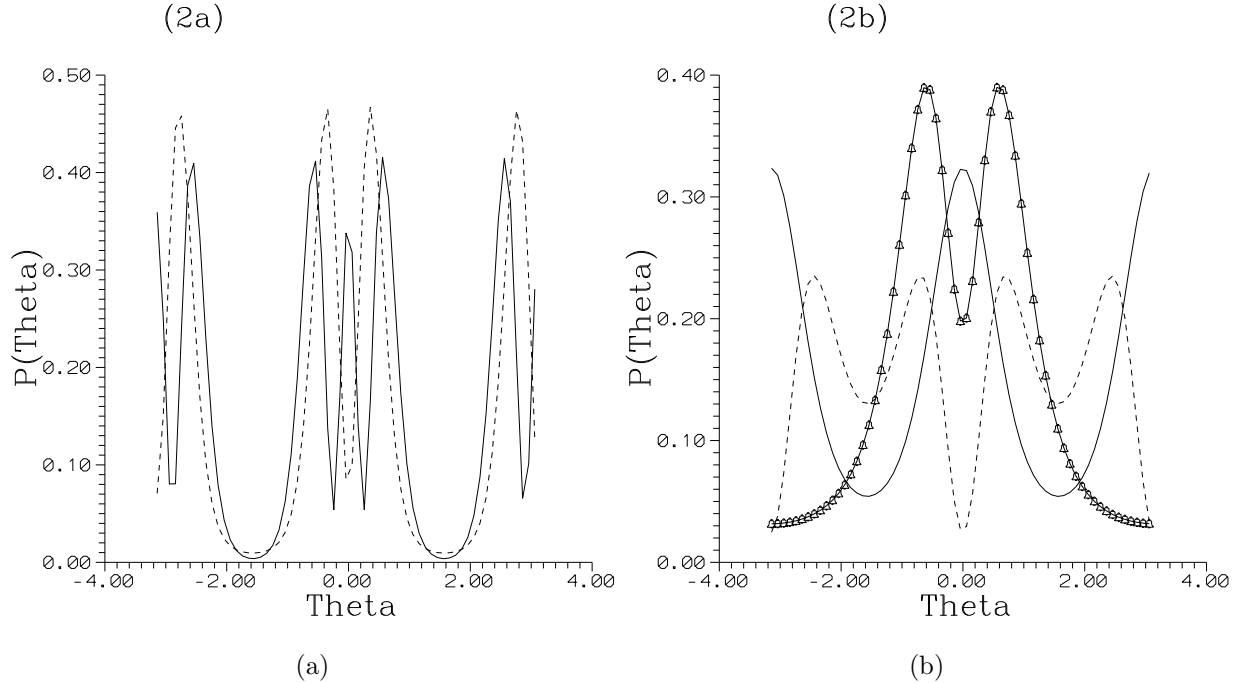


FIG. 2: Phase distribution $P(\Theta)$ for superposition of displaced number states and for a) $\alpha = 2, \epsilon = 1, n = 1$ (dashed curve), and 2 (solid curve); b) $\alpha = 1, n = 1$ and $\epsilon = 0$ (bell-centered curve), 1 (solid curve), and -1 (dashed curve).

Before discussing the phase properties of the displaced and squeezed number states it is reasonable to remind the behaviour of the well known squeezed states. As known for squeezed states with non-zero displacement coherent amplitude, the phase distribution exhibits the bifurcation phenomenon. In this phenomenon the single peak structure of the coherent

component is evolved into two peaks structure with respect to both α (for large fixed value of squeezing parameter r) and r (when α takes fixed value) [11]. This phenomenon has been recognized as a result of the competition between the two peaks structure of the squeezed vacuum state and the single-peak structure of the coherent state. For squeezed and displaced number states such phenomenon cannot occur due to the effect of the Fock state which replaces the initial peak ($r = 0$) for coherent state by a multi-peak structure, i.e. by $n + 1$ peaks (see Fig. 3a for shown values of the parameters). From this figure one can observe that there is a three-peak structure corresponding to the case $n = 2$ of displaced Fock state. The height of the central peak (i.e. at $\Theta = 0$) is almost the same and equals $(1/2\pi)|\sum_{m=0}^{\infty} C_m|^2$. That is the central value of the phase distribution $P(\Theta = 0)$ is insensitive to squeezing provided that r is finite. However, the lateral peaks undergo phase squeezing as r increases, i.e. the peaks become narrower.

Now we can investigate the behaviour of the superposition of displaced and squeezed number states (see Figs. 3b-d for shown parameters). Figs. 3b and 3c are given for the first regime for even- and odd-cases, respectively. From Fig. 3b we can see that the initial oscillations are increased compared with Fig. 3a as a result of the interference in phase space. Further, the initial lateral peaks can evolve in the course of increasing r to provide bifurcation shape, i.e. the distribution curve undergoes a transition from single- to a double-peaked form with increasing r ; however, the central peak is almost unchanged with increasing r . This peak splitting is connected with the squeezed states [11]. Further, as the number of peaks increases for $r > 0$, the distribution becomes more and more narrower. On the other hand, the phase distribution of the odd-case is quite different as we have shown before where the initial peaks are not significantly changing as in the even-case. More precisely, initial distribution becomes broader for a while and suddenly (at $r \simeq 0.5$) breaks off to start to be a narrower distribution for later r . It is clear that in this regime the state (1) becomes more and more nonclassical [15] and the even- and odd-cases are distinguishable. Further, the phase distribution of the even-case is more sensitive with respect to squeezing than the odd-case. Nevertheless, for the second regime $\alpha \gg 1$ we noted that the phase distribution carries at least the same initial information regardless of the value of r (see Fig. 3d for the shown values of the parameters).

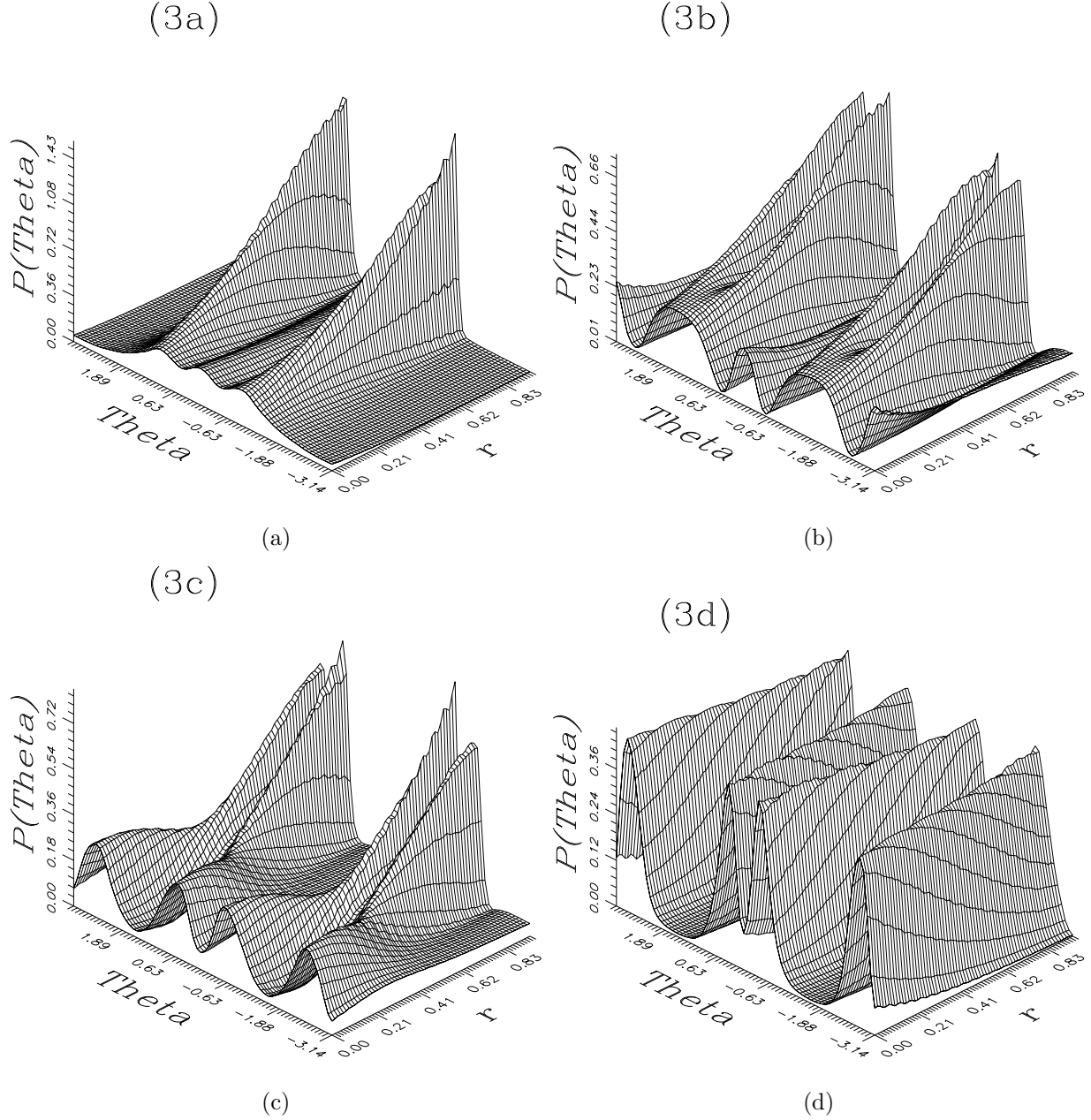


FIG. 3: Phase distribution $P(\Theta)$ for superposition of displaced and squeezed number states for $(\epsilon, \alpha, n) = (0, 1, 2), (1, 1, 2), (-1, 1, 2)$ and $(1, 2, 2)$ corresponding to the cases a,b,c and d, respectively.

B. Variance, amplitude and phase squeezing

In this subsection we investigate the behaviour of the phase variance, and amplitude and phase fluctuations following (26), (18) and (19), respectively. We start our discussion by analysing the behaviour of the superposition of displaced number states. For this purpose

Figs. 4a and 4b are shown for the phase variance, and amplitude and phase fluctuations, respectively. It is seen that in general the phase variance starts from the value $\pi^2/3$ (the vacuum state value) and returns back to it when α is large, but through different routes. To be more specific, the phase variance of displaced number states starts from the value for the vacuum, goes to a minimum, and then comes again to $\pi^2/3$. However, the behaviour of the superposition of displaced number states takes different ways to arrive at the same result, i.e. it starts from $\pi^2/3$ as before, goes to the maximum value and eventually comes back to the value of vacuum. The comparison of the two cases shows the role of the quantum mechanical interference between state components. Further, as n increases, the oscillations in the variance become more pronounced. Comparison of the behaviour of even- and odd-cases shows that they are different only over the initial short interval of α , i.e. when α is small, and this agrees with what we have discussed earlier. So we can conclude that for intensities high enough of the coherent field, the variance of the phase is approximately randomized. The route to this randomization is dependent on the choice of ϵ . With respect to the amplitude and phase fluctuations, we can note from (17) that these quantities depend not only on the intensity of the field, but also on the choice of the reference angle Θ_0 . We have chosen here $\Theta_0 = -\pi$, where the mean value $|\langle[\hat{N}, \hat{\Phi}_\theta]\rangle|$ of squeezed displaced number state approaches unity. Fig. 4b has been obtained to illustrate the parameters S_N and S_θ which provide information about the degree of squeezing in \hat{n} and $\hat{\Phi}_\theta$. One can observe from this figure that when $\epsilon = 0$ (displaced number state) and $\alpha \rightarrow 0$, the parameter S_N tends to -1 , which means that the number state is 100% squeezed with respect to the operator \hat{n} . This situation is expected since $\langle(\Delta\hat{n})^2\rangle = 0$ for the number state. Further, the larger the number of quanta is the shorter is the interval over which S_N is squeezed. Also when $\alpha \gg 1$, squeezing in S_θ is remarkable, whereas S_N becomes unsqueezed. This result can be deduced from the behaviour of the phase variance (see Fig. 4a), where $\langle(\Delta\hat{\Phi})^2\rangle \simeq 0$ at $\alpha \simeq 4$ and this should be connected with maximum squeezing in S_θ at this point. Such behaviour of S_N and S_θ confirms the fact that the number of photons and phase are conjugate quantities in this approach. As is known displaced number states are not minimum uncertainty states and the variances for the quadrature operators never go below the standard quantum limit. Moreover, they may exhibit sub-Poissonian statistics for the range $\alpha^2 \leq 1/2$ [13]. However, there is no relation between the sub-Poissonian statistics and the fluctuation in the amplitude or the phase. This fact has been shown before for the down-conversion process with quantum

pump where the signal mode can exhibit amplitude squeezing and at the same time it is super-Poissonian [27].

We proceed by discussing the behaviour of the superposition states (long-dashed and circle-centered curves) in Fig. 4b where the interference in phase space starts to play a role. We noted (from our numerical analysis) that only the even case can provide squeezing in S_N with maximum value at the origin and squeezing interval larger than that discussed before. Indeed, this maximum value is related also to that of number state at $\alpha = 0$. It should be stressed here that $\alpha = 0$ for the odd-case may lead to a singularity. As we mentioned before the superposition of displaced number states can exhibit strong sub-Poissonian character as well as quadrature squeezing [23]. Now we can illustrate the role of the squeezing in the superimposed displaced number states optical cavity with respect to variance, amplitude and phase fluctuations. It is obvious, when squeezing is considered, that the initial value (at $\alpha = 0$) of the phase variance is shifted since we have initially squeezed number state which is providing phase information. However, when α is large and r is finite or also r is large, it can be proved simply that the coefficient $C_m(r, \alpha, n, \epsilon)$ vanishes and consequently the phase variance tends to $\pi^2/3$ (becomes randomized). Moreover, the routs are here similar to those of Fig. 4a. On the other hand, squeezing could be seen only in S_θ for $\epsilon = 0$ (see squared-centered curve in Fig. 4b). Furthermore, comparison of the short-bell-centered curve (of displaced Fock state) and squared-centered curve reveals that squeezing parameter reduces the amount of squeezing in S_θ , too. This means that the superposition of displaced and squeezed number states which provide quadratures squeezing has less information about amplitude and phase fluctuations.

IV. CONCLUSIONS

We have discussed the phase properties of the superposition of squeezed and displaced number states from the point of view of the Pegg-Barnett Hermitian phase formalism. Moreover, the phase variance, the photon number and phase fluctuations have been discussed, too. The results have been explored graphically.

We have shown that there are two regimes controlling the behaviour of the phase for such superposition depending on whether the superposition is macroscopic ($\alpha \gg 1$) or microscopic ($\alpha \leq 1$). This fact has been illuminated by means of the Wigner function. In

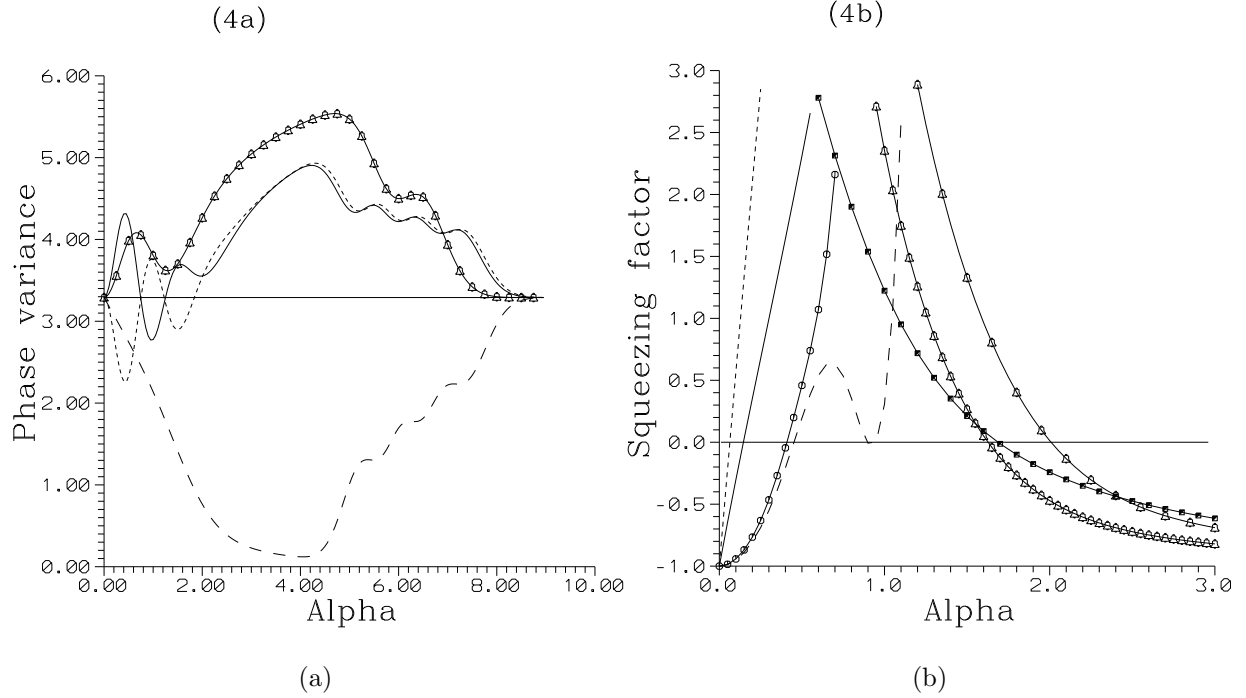


FIG. 4: a) Phase variance of a superposition of displaced number states against α for $(\epsilon, n) = (0, 3)$ (long-dashed curve), $(1, 1)$ (bell-centered curve), $(1, 3)$ (solid curve) and $(-1, 3)$ (short-dashed curve). The solid straight line is corresponding to the phase variance of vacuum. b) Amplitude and phase fluctuations of a superposition of displaced and squeezed number states against α for $(\epsilon, n, r) = (0, 1, 0)$ (S_N solid curve and S_θ short-bell-centered curve), $(0, 2, 0)$ (S_N short-dashed curve and S_θ long-bell-centered curve), $(1, 1, 0)$ (S_N long-dashed curve), $(1, 2, 0)$ (S_N circle-centered curve) and $(1, 2, 0.5)$ (S_θ squared-centered curve). The solid straight line is the bound of squeezing.

the first regime, the even- and odd-cases (i.e. $\epsilon = 1$ and -1) give similar behaviours and the structure of the density matrix is remarkable in figures. All these facts are washed out in the second regime where the behaviour becomes irregular, however, smooth. In general we noted that the higher the number of quanta is, the more peaks the distribution possesses. Influence of the squeezing parameter could be recognized in the second regime where the distribution exhibits peak-splitting and peak-narrowing and this is in contrast with the first regime.

For the phase variance we conclude that it asymptotically goes to the value $\pi^2/3$ of the uniform distribution when either α or r is large, but through different routes. We have shown also that this superposition can exhibit photon number fluctuations and phase fluctuations.

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